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WHITEHEAD DOUBLE AND MILNOR INVARIANTS

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Abstract

We consider the operation of Whitehead double on a component of a link and study the behavior of Milnor invariants under this operation. We show that this operation turns a link whose Milnor invariants of length $\leq k$ are all zero into a link with vanishing Milnor invariants of length $\leq 2k+1$, and we provide formulae for the first non-vanishing ones. As a consequence, we obtain statements relating the notions of link-homotopy and self Δ-equivalence via the Whitehead double operation. By using our result, we show that a Brunnian link L is link-homotopic to the unlink if and only if the link L with a single component Whitehead doubled is self Δ-equivalent to the unlink.

1. Introduction

In this paper, we consider the operation of Whitehead double, more generally of Whitehead n-double, on a component of a link, and we study the behavior of Milnor invariants under this operation. Milnor invariants $\overline{\mu}_L(I)$ of an m-component link L, where $I = i_1 i_2 \cdots i_k$ with $1 \leq i_j \leq m$, can be thought of as some sort of "higher order linking number" of the link. See Section 2 for a definition.

A typical example is the Whitehead link, which is a Whitehead double of the Hopf link. The linking number of the Hopf link (which coincides with Milnor invariant $\overline{\mu}(12)$) is ± 1, whereas the Whitehead link has linking number 0. On the other hand, the Whitehead link has some nontrivial higher order Milnor invariants: its Sato–Levine invariant for instance, which is equal to $-\overline{\mu}(1122)$, is ± 1. Our main result, stated below, generalizes this observation.

Let K be a component of a link L in S^3, regarded as $h(\{\mathbf{0}\} \times S^1)$ for some embedding $h\colon D^2 \times S^1 \to S^3 \setminus (L \setminus K)$, such that K and $h((0,1) \times S^1)$ have linking number zero. Let n be a (nonzero) integer. Consider in the solid torus $T = D^2 \times S^1$ the knot $\mathcal{W}_n$ depicted in Fig. 1.1. The knot $h(\mathcal{W}_n)$ is called the *Whitehead n-double of K*, and it is denoted by $W_n(K)$.

Given an m-component link $L = K_1 \cup \cdots \cup K_m$ in S^3, we denote by $W_n^i(L)$ the link $(L \setminus K_i) \cup W_n(K_i)$ obtained by Whitehead n-double on the i^{th} component of L.

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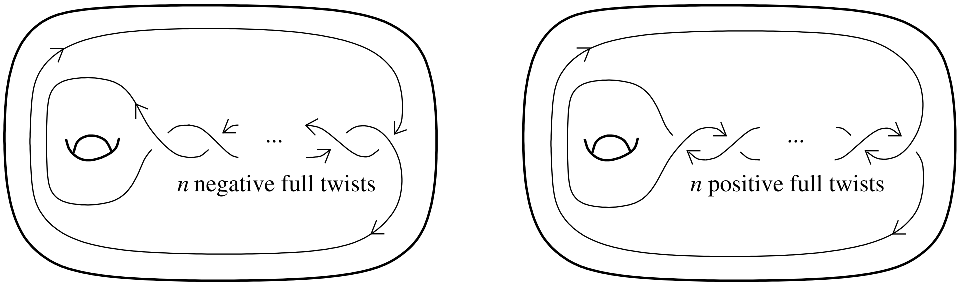

Fig. 1.1. The knot $\mathcal{W}_n$ for $n < 0$ and $n > 0$ respectively.

Note that the case $n = \pm 1$ coincides with the usual notion of (positive or negative) Whitehead double.

Theorem 1.1. *Let L be an m-component link in S^3, and let n $(\neq 0)$ be an integer. If all Milnor invariants $\overline{\mu}_L(Ji)$ of L of length $|Ji| \leq k$ are zero $(k \geq 1)$, then all Milnor invariants $\overline{\mu}_{W_n^i(L)}(Ii)$ of $W_n^i(L)$ of length $|Ii| \leq 2k+1$ are zero. Moreover, if $\overline{\mu}_L(Pi) \neq 0$, $\overline{\mu}_L(Qi) \neq 0$ with $P = p_1 p_2 \cdots p_k$, $Q = q_1 q_2 \cdots q_k$ (possibly $P = Q$) such that $p_j \neq i$, $q_j \neq i$ for all $1 \leq j \leq k$, then we have the following formulae for the first non-vanishing Milnor invariants of $W_n^i(L)$*

$$\begin{cases} \overline{\mu}_{W_n^i(L)}(PiQi) = 2n\overline{\mu}_L(Pi)\overline{\mu}_L(Qi), \\ \overline{\mu}_{W_n^i(L)}(PQii) = -n\overline{\mu}_L(Pi)\overline{\mu}_L(Qi). \end{cases}$$

REMARK 1.2. In the case of a 2-component link, the formulae given in Theorem 1.1 for the first nonvanishing Milnor invariants of $W_n^i(L)$ provide, as an immediate corollary, a generalization of a result of Shibuya and the second author [14] as follows: Let $L = K_1 \cup K_2$ in S^3. Let $n \neq 0$ be an integer, and let $W_n(L)$ be obtained by Whitehead n-double on a component of L. Then the Sato–Levine invariant β_2 of $W_n(L)$ satisfies

$$\beta_2(W_n(L)) = n(lk(K_1, K_2))^2.$$

(Note that the Sato–Levine invariant of $W_n(L)$ is well-defined, as Theorem 1.1 ensures that the link has zero linking number.)

Recall that two links are *link-homotopic* if they are related by a sequence of ambient isotopies and *self crossing changes*, which are crossing changes involving two strands of the same component, see the left-hand side of Fig. 1.2. In particular, a link is called *link-homotopically trivial* if it is link-homotopic to the unlink. It has long been known that Milnor invariants with no repeating indices are invariants of link-homotopy [5]. Like crossing change, the Δ-move is an unknotting operation [6]. Here we consider

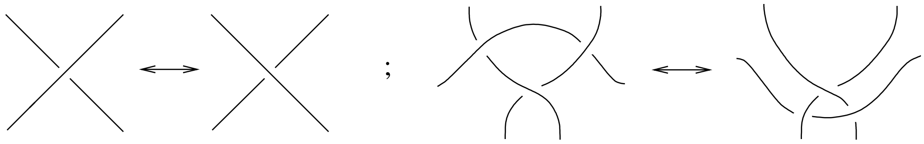

Fig. 1.2. A crossing change and a Δ-move.

the notion of *self Δ-move* for links, which is a local move as illustrated in the right-hand side of Fig. 1.2 involving three strands of the same component. Two links are *self Δ-equivalent* if they are related by a finite sequence of ambient isotopies and self Δ-moves. Self Δ-equivalence is a generalized link-homotopy, i.e., self Δ-equivalence implies link-homotopy. The self Δ-equivalence was introduced by Shibuya [10, 11], and was subsequently studied by various authors [2, 7, 8, 9, 13, 14, 16]. A link is *self Δ-trivial* if it is self Δ-equivalent to the unlink.

The following is a consequence of our main result.

Corollary 1.3. *Let L be an m-component link in S^3 which is not link-homotopically trivial. Then, for any n $(\neq 0)$ and i $(1 \leq i \leq m)$, $W_n^i(L)$ is not self Δ-trivial.*

Recall now that a link L is Brunnian if all proper sublinks of L are trivial. The next result shows that the converse of Corollary 1.3 also holds for Brunnian links.

Theorem 1.4. *Let L be an m-component Brunnian link in S^3. Let n $(\neq 0)$ and i $(1 \leq i \leq m)$ be integers. Then L is link-homotopically trivial if and only if $W_n^i(L)$ is self Δ-trivial.*

Observe that an m-component Brunnian link always has vanishing Milnor invariants of length $\leq m-1$ since these are Milnor invariants of sublinks of a Brunnian link, which are trivial links. So Theorem 1.1 implies that all Milnor invariants of $W_n^i(L)$ of length $\leq 2m-1$ are zero for any choice of $1 \leq i \leq m$ and n $(\neq 0)$. In other words, for m-component Brunnian links, Whitehead doubling kills all Milnor invariants of length $\leq 2m-1$. It follows from a more general result (stated and proved in Section 4) that an additional Whitehead doubling, on either the same or another component of the link, actually kills *all* Milnor invariants, as the resulting link is always a boundary link, see Corollary 4.2.

The rest of the paper is organized as follows. In Section 2 we recall the definition of Milnor invariants and prove Theorem 1.1. In Section 3 we prove the two statements relating Whitehead doubling and self Δ-equivalence, namely Corollary 1.3 and Theorem 1.4. In Section 4 we consider more general satellite constructions, involving a knot which is null-homologous in the solid torus. When applied twice to a Brunnian link, such a construction always yields a boundary link.

2. Milnor invariants

J. Milnor defined in [4, 5] a family of invariants of oriented, ordered links in S^3, known as Milnor's $\overline{\mu}$-invariants.

Given an m-component link L in S^3, denote by $\pi(L)$ the fundamental group of $S^3 \setminus L$, and by $\pi_q(L)$ the q^{th} subgroup of the lower central series of $\pi(L)$. We have a presentation of $\pi(L)/\pi_q(L)$ with m generators, given by a meridian α_i of the i^{th} component of L. So for $1 \le i \le m$, the longitude l_i of the i^{th} component of L is expressed modulo $\pi_q(L)$ as a word in the α_i's (abusing notations, we still denote this word by l_i).

The *Magnus expansion* $E(l_i)$ of l_i is the formal power series in non-commuting variables $X_1, \ldots, X_m$ obtained by substituting $1 + X_j$ for α_j and $1 - X_j + X_j^2 - X_j^3 + \cdots$ for α_j^{-1}, $1 \le j \le m$.

Let $I = i_1 i_2 \cdots i_{k-1} j$ be a multi-index (i.e., a sequence of possibly repeating indices) among $\{1, \ldots, m\}$. Denote by $\mu_L(I)$ the coefficient of $X_{i_1} \cdots X_{i_{k-1}}$ in the Magnus expansion $E(l_j)$. *Milnor invariant* $\overline{\mu}_L(I)$ is the residue class of $\mu_L(I)$ modulo the greatest common divisor of all $\mu_L(J)$ such that J is obtained from I by removing at least one index, and permutating the remaining indices cyclically. We call $|I| = k$ the *length* of Milnor invariant $\overline{\mu}_L(I)$.

The indeterminacy comes from the choice of the meridians α_i or, equivalently, from the indeterminacy of representing the link as the closure of a string link [3].

Proof of Theorem 1.1. Without loss of generality, we may suppose that $i = m$. We give the proof of the case $n < 0$. The case $n > 0$ is strictly similar and we omit it.

We denote by $\alpha_1, \ldots, \alpha_{m-1}$, α_m and a meridians of $K_1, \ldots, K_{m-1}$, K_m and $W_n(K_m)$ respectively, such that $\alpha_1, \ldots, \alpha_m$ generate $\pi(L)/\pi_q(L)$ and $\alpha_1, \ldots, \alpha_{m-1}, a$ generate $\pi(W_n^m(L))/\pi_q(W_n^m(L))$.

The Magnus expansion of the longitude $l_m \in \pi(L)/\pi_q(L)$ of K_m, written as a word in $\alpha_1, \ldots, \alpha_m$, has the form

$$E(l_m) = 1 + \sum \mu_L(i_1 \cdots i_r, m) X_{i_1} \cdots X_{i_r} = 1 + f(X_1, \ldots, X_m),$$

where $E(\alpha_i) = 1 + X_i$ for all $1 \le i \le m$.

Now consider the Whitehead n-double of K_m, and consider $2n+1$ elements $a_0, a_1, \ldots, a_{2n}$ of $S^3 \setminus W_n^m(L)$ as represented in Fig. 2.1. Let $\phi(l_m) = l$, where $\phi\colon \pi(L)/\pi_q(L) \to \pi(W_n^m(L))/\pi_q(W_n^m(L))$ is the natural map that maps α_i to itself $(1 \le i \le m-1)$ and maps α_m to $a_{2n}^{-1} a$. (Abusing notation, we still denote by a_i, $0 \le i \le 2n$, the corresponding elements in $\pi(W_n^m(L))/\pi_q(W_n^m(L))$.)

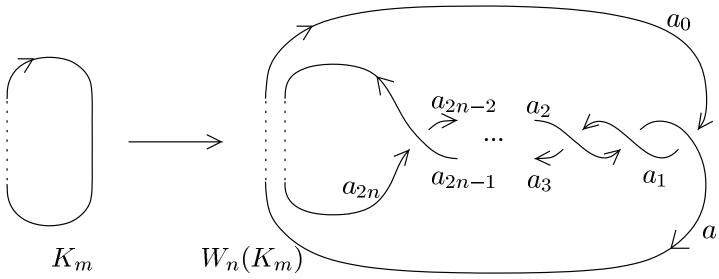

Fig. 2.1. The Whitehead n-double of K_m for $n < 0$.

It follows from repeated uses of Wirtinger relations that

$$\begin{cases} a_0 = l^{-1}al, & \\ a_{2r} = R^r a R^{-r}, & \text{for all} \quad r \geq 1, \\ a_{2r+1} = R^r a R^{-(r+1)}, & \text{for all} \quad r \geq 0 \end{cases}$$

where $R = al^{-1}a^{-1}l$. In particular we have that

$$\phi(\alpha_m) = a_{2n}^{-1}a = R^n a^{-1} R^{-n} a.$$

Let $E(a) = 1 + X$ denote the Magnus expansion of a. Observe that

$$\begin{aligned} E(R) = E(al^{-1}a^{-1}l) &= (1 + X)E(l^{-1})(1 - X)E(l) + \mathcal{O}_X(2) \\ &= 1 + X - E(l^{-1})XE(l) + \mathcal{O}_X(2), \end{aligned}$$

and

$$\begin{aligned} E(R^{-1}) = E(l^{-1}ala^{-1}) &= E(l^{-1})(1 + X)E(l)(1 - X) + \mathcal{O}_X(2) \\ &= 1 - X + E(l^{-1})XE(l) + \mathcal{O}_X(2), \end{aligned}$$

where $\mathcal{O}_X(2)$ denotes terms which contain X at least 2 times. So we have

$$\begin{aligned} E(\phi(\alpha_m)) &= (1 + X - E(l^{-1})XE(l))^n(1 - X) \\ &\quad \times (1 - X + E(l^{-1})XE(l))^n(1 + X) + \mathcal{O}_X(2) \\ &= (1 + nX - nE(l^{-1})XE(l))(1 - X) \\ &\quad \times (1 - nX + nE(l^{-1})XE(l))(1 + X) + \mathcal{O}_X(2) \\ &= 1 + \mathcal{O}_X(2). \end{aligned}$$

This implies that

$$\begin{aligned} E(l) &= 1 + f(X_1, \dots, X_{m-1}, \mathcal{O}_X(2)) \\ &= 1 + f_1(X_1, \dots, X_{m-1}) + f_2(X_1, \dots, X_{m-1}, X), \end{aligned}$$

where

$$f_1(X_1, \dots, X_{m-1}) = f(X_1, \dots, X_{m-1}, 0) \in \mathcal{O}(k)$$

and

$$f_2(X_1, \dots, X_{m-1}, X) = f(X_1, \dots, X_{m-1}, \mathcal{O}_X(2)) - f_1(X_1, \dots, X_{m-1}) \in \mathcal{O}(k+1),$$

and $\mathcal{O}(u)$ denotes terms of degree at least u (the degree of a monomial in the X_j is simply defined by the sum of the powers). Similarly we have

$$\begin{aligned} E(l^{-1}) &= 1 + g(X_1, \dots, X_{m-1}, \mathcal{O}_X(2)) \\ &= 1 + g_1(X_1, \dots, X_{m-1}) + g_2(X_1, \dots, X_{m-1}, X), \end{aligned}$$

where $g_1(X_1, \dots, X_{m-1}) \in \mathcal{O}(k)$ and $g_2(X_1, \dots, X_{m-1}, X) \in \mathcal{O}(k+1)$.

Let f_1, f_2, g_1, g_2 denote $f_1(X_1, \dots, X_{m-1})$, $f_2(X_1, \dots, X_{m-1}, X)$, $g_1(X_1, \dots, X_{m-1})$, $g_2(X_1, \dots, X_{m-1}, X)$ respectively, and set $f = f_1 + f_2$ and $g = g_1 + g_2$. Set $E(a^{-1}) = 1 - X + X^2 - X^3 + \cdots = 1 + Y$. Note that $(1+f)(1+g) = (1+g)(1+f) = 1$ and $(1+X)(1+Y) = (1+Y)(1+X) = 1$, hence $f + g = -fg = -gf \in \mathcal{O}(2k)$ and $X + Y = -XY = -YX$. One can check, by induction, that

$$\begin{cases} E(R^n) = 1 + n(gY - Xf + XgY + gYf) + \mathcal{O}(2k+2), \\ E(R^{-n}) = 1 + n(Xf - gY + XfY + gXf) + \mathcal{O}(2k+2), \\ E((a^{-1}R)^n) = (1+Y)^n + (1+Y)^n f - f(1+Y)^n + n(gYf - fgY) + \mathcal{O}(2k+2). \end{cases}$$

Since the preferred longitude L_m of $W_n^m(K_m)$ is presented in $\pi(W_n^m(L))/\pi_q(W_n^m(L))$ by the word

$$L_m = la^{-1}a_2^{-1} \cdots a_{2n-2}^{-1} l^{-1} a_{2n-1}^{-1} a_{2n-3}^{-1} a_3^{-1} a_1^{-1} a^{2n} = l(a^{-1}R)^n R^{-n} l^{-1} R^n a^n,$$

we have

$$\begin{aligned} E(L_m) &= (1+f)[(1+Y)^n + (1+Y)^n f - f(1+Y)^n + n(gYf - fgY)] \\ &\qquad \times [1 + n(Xf - gY + XfY + gXf)](1+g) \\ &\qquad \times [1 + n(gY - Xf + XgY + gYf)](1+X)^n \\ &= [(1+Y)^n + n(2fXf - f^2X - Xf^2)](1+X)^n + \mathcal{O}(2k+2) \\ &= 1 + n(2fXf - ffX - Xff) + \mathcal{O}(2k+2). \end{aligned}$$

Because $f \in \mathcal{O}(k)$, the first non-trivial terms in the Magnus expansion $E(L_m)$ are of degree $2k+1$. It follows that all Milnor invariants $\overline{\mu}_{W_n^m(L)}(Im)$ of length $|Im| \leq 2k+1$ of $W_n^m(L)$ are zero.

Moreover, we actually have

$$E(L_m) = 1 + n(2 f_1 X f_1 - f_1 f_1 X - X f_1 f_1) + \mathcal{O}(2k+2).$$

So if $\overline{\mu}_L(Pm) \neq 0$, $\overline{\mu}_L(Qm) \neq 0$ for some multi-indices $P = p_1 \cdots p_k$, $Q = q_1 \cdots q_k$ ($P \neq Q$) with $p_j \neq m$, $q_j \neq m$ for all $1 \leq j \leq k$, then

$$f_1 = \overline{\mu}_L(Pm) X_{p_1} \cdots X_{p_k} + \overline{\mu}_L(Qm) X_{q_1} \cdots X_{q_k} + \mathcal{O}(k),$$

and it follows from the above formula that

$$\begin{aligned}
E(L_m) = 1 &+ 2n\overline{\mu}_L(Pm)\overline{\mu}_L(Pm) X_{p_1} \cdots X_{p_k} X X_{p_1} \cdots X_{p_k} \\
&+ 2n\overline{\mu}_L(Pm)\overline{\mu}_L(Qm) X_{p_1} \cdots X_{p_k} X X_{q_1} \cdots X_{q_k} \\
&+ 2n\overline{\mu}_L(Qm)\overline{\mu}_L(Pm) X_{q_1} \cdots X_{q_k} X X_{p_1} \cdots X_{p_k} \\
&+ 2n\overline{\mu}_L(Qm)\overline{\mu}_L(Qm) X_{q_1} \cdots X_{q_k} X X_{q_1} \cdots X_{q_k} \\
&- n\overline{\mu}_L(Pm)\overline{\mu}_L(Pm) X_{p_1} \cdots X_{p_k} X_{p_1} \cdots X_{p_k} X \\
&- n\overline{\mu}_L(Pm)\overline{\mu}_L(Qm) X_{p_1} \cdots X_{p_k} X_{q_1} \cdots X_{q_k} X \\
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&- n\overline{\mu}_L(Qm)\overline{\mu}_L(Qm) X X_{q_1} \cdots X_{q_k} X_{q_1} \cdots X_{q_k} + \mathcal{O}(2k+1)
\end{aligned}$$

which implies the desired formulae for the first nonvanishing Milnor invariants of $W_n^m(L)$. □

REMARK 2.1. One may wonder what happens when we consider, in the definition of a Whitehead n-double, an odd number of half-twists in place of n full twists. For a link L, denote by $W^i_{\text{odd}}(L)$ any link obtained by such a satellite construction with an odd number of half-twists on the i^{th} component of L. Then we can prove the following: If all Milnor invariants of L with length $\leq k$ vanish, then for any multi-index Ii with $|Ii| \leq k+1$, $\overline{\mu}_{W^i_{\text{odd}}(L)}(Ii) = 2^{r_i+1}\overline{\mu}_L(Ii)$, where r_i is the number of times that the index i appears in I.

3. On self Δ-equivalence

In this section we provide the proofs for Corollary 1.3 and Theorem 1.4.

We need the following additional notation. Given a multi-index I, we denote by $r(I)$ the maximum number of times that any index appears in I. For example, $r(1123) = 2$ and $r(1233212) = 3$.

Proof of Corollary 1.3. Let L be an m-component link which is not link-homotopically trivial. Then by [4] there exists some multi-index $I = i_1 \cdots i_p$ with $r(I) = 1$ such that $\overline{\mu}_L(I) \neq 0$ and $\overline{\mu}_L(J) = 0$ for all multi-index J with length $|J| < |I|$ and $r(J) = 1$.

Let n ($\neq 0$) and i ($1 \leq i \leq m$) be integers. If I does not contain i, then $\overline{\mu}_{W^i_n(L)}(I) = \overline{\mu}_L(I) \neq 0$. So $W^i_n(L)$ is not link-homotopically trivial. Hence $W^i_n(L)$ is not self Δ-trivial. Suppose that I contains i. By "cyclic symmetry" ([5, Theorem 6]), we may assume that $i_p = i$. By Theorem 1.1, the link $W^i_n(L)$ thus satisfies $\overline{\mu}_{W^i_n(L)}(M) \neq 0$ for some multi-index M with $r(M) \leq 2$. Since Milnor invariants with $r \leq 2$ are self Δ-equivalence invariants [1], $W^i_n(L)$ is not self Δ-trivial. □

Proof of Theorem 1.4. Let L be an m-component Brunnian link. Let n ($\neq 0$) and i ($1 \leq i \leq m$) be integers. By Corollary 1.3 we already know that L is link-homotopically trivial if $W^i_n(L)$ is self Δ-trivial. Let us prove that the converse is also true.

The link L being Brunnian, $\overline{\mu}_L(I) = 0$ if I does not contain an index in $\{1, \dots, m\}$. Moreover, if L is link-homotopically trivial, then $\overline{\mu}_L(I) = 0$ for any I with $r(I) = 1$. In particular $\overline{\mu}_L(I) = 0$ for all $|I| \leq m$, and by Theorem 1.1 the link $W^i_n(L)$ thus satisfies $\overline{\mu}_{W^i_n(L)}(I) = 0$ for all $|I| \leq 2m+1$. This implies that $\overline{\mu}_{W^i_n(L)}(I) = 0$ for any multi-index I with $r(I) \leq 2$. By [16, Corollary 1.5], we have that $W^i_n(L)$ is self Δ-trivial. □

4. From Brunnian links to boundary links

4.1. Boundary links from satellite construction. In this section we consider a more general satellite construction.

Let $L = K_1 \cup \cdots \cup K_m$ be an m-component link in S^3, and let $h_i \colon D^2 \times S^1 \to S^3$ be an embedding such that $h_i(\{\mathbf{0}\} \times S^1)$ is the i^{th} component K_i of L (as in the introduction, we assume that K_i and $h((0, 1) \times S^1)$ have linking number zero). Now, instead of the knot $\mathcal{W}_n$ depicted in Fig. 1.1, consider in the solid torus $T = D^2 \times S^1$ a fixed knot $\mathcal{K}$ which is null-homologous in T. Denote by $W^i_{\mathcal{K}}(L)$ the link $(L \setminus K_i) \cup h_i(\mathcal{K})$. We have the following result.

Theorem 4.1. *Let $L = K_1 \cup \cdots \cup K_m$ be an m-component link in S^3, and let $\mathcal{K}$, $\mathcal{K}'$ be two null-homologous knots in the solid torus T. Then*

(i) *If $L \setminus K_i$ is a boundary link, then $W^i_{\mathcal{K}}(W^i_{\mathcal{K}'}(L))$ is a boundary link.*

(ii) *If $L \setminus (K_i \cup K_j)$ is a boundary link and $K_i \cup K_j$ is null-homotopic in $S^3 \setminus (L \setminus (K_i \cup K_j))$, then $W^i_{\mathcal{K}}(W^j_{\mathcal{K}'}(L))$ is a boundary link.*

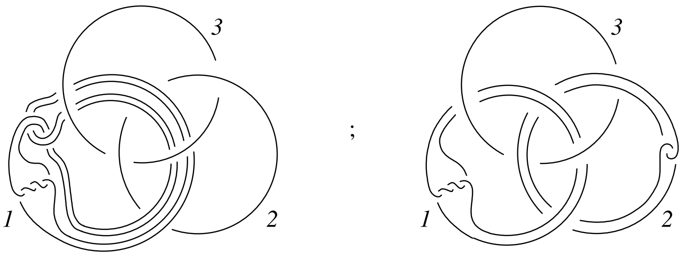

Fig. 4.1. The boundary links $W^{1,1}_{-4,2}(B)$ and $W^{1,2}_{-4,2}(B)$.

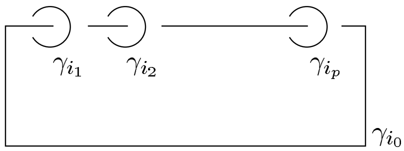

Fig. 4.2. The link L_i.

Note that in particular a Brunnian link L always satisfies the conditions in (i) and (ii). It follows that a link obtained from a Brunnian link by taking twice Whitehead double (on either the same or another component of the link) kills *all* Milnor invariants.

Corollary 4.2. *Let L be an m-component Brunnian link in S^3. Let p, q $(pq \neq 0)$ and $i, j \in \{1, \dots, m\}$ (possibly equal) be integers. Then the link $W^{i,j}_{p,q}(L)$, obtained by respectively Whitehead p-double and Whitehead q-double on the i^{th} and j^{th} components of L, is a boundary link.*

Fig. 4.1 below illustrates this result in the case of the Borromean rings B.

4.2. Proof of Theorem 4.1. Before proving Theorem 4.1, we will introduce the notion of band presentation of a link.

Let $L_i = \gamma_{i0} \cup \gamma_{i1} \cup \gamma_{i2} \cup \cdots \cup \gamma_{ip_i}$ be a link as illustrated in Fig. 4.2. Let $L_1 \cup \cdots \cup L_m$ be a split union of the links $L_1, \dots, L_m$, and let $\Delta = \bigcup \Delta_{ij}$ be a disjoint union of disks Δ_{ij} $(1 \leq i \leq m;\ 1 \leq j \leq p_i)$ such that $\partial \Delta_{ij} = \gamma_{ij}$ and $\Delta_{ij} \cap \left(\bigcup_k \gamma_{k0}\right) = \Delta_{ij} \cap \gamma_{i0}$ consists of a single point. It is known [15] that an m-component link L in a 3-manifold M which is null-homotopic in M can be expressed as a band sum of $L_1 \cup \cdots \cup L_m$, which is contained in a 3-ball in M, along mutually disjoint bands b_{ij} $(1 \leq i \leq m;\ 1 \leq j \leq p_i)$, disjoint from $\operatorname{int} \Delta$, such that b_{ij} connect γ_{ij} and $\left(\bigcup_k \gamma_{k0}\right)$.[1] This presentation is called a *band presentation* of L, and $L_1 \cup \cdots \cup L_m$ is called the *base link*.

[1]The result is given in [15] for *knots* in S^3, but it can be easily extended to the link case.

Proof of Theorem 4.1. (i) We may suppose that $i = m$ without loss of generality. Since $K_1 \cup \cdots \cup K_{m-1}$ is a boundary link, it bounds a disjoint union of surfaces $E = E_1 \cup \cdots \cup E_{m-1}$. Denote by $W_{\mathcal{K}'}(K_m)$ the m^{th} component of $W^m_{\mathcal{K}'}(L)$. Since $W_{\mathcal{K}'}(K_m)$ is null-homologous in $h_m(D^2 \times S^1)$, it is null-homotopic in $S^3 \setminus (L \setminus K_m)$. Hence there is a band presentation of $W_{\mathcal{K}'}(K_m)$ such that the base link is disjoint from E and such that the intersections of each band and E are ribbon singularities. So $W_{\mathcal{K}'}(K_m) \cap E$ is a union of copies of S^0, which are the endpoints of these ribbon singularities. By tubing the surfaces E_i suitably at these endpoints, we obtain a union of mutually disjoint surfaces $F_1, \ldots, F_{m-1}$ such that $F_i = \partial K_i$ and $F_i \cap W_{\mathcal{K}'}(K_m) = \emptyset$ for all $1 \leq i \leq m-1$. Since the m^{th} component of $W^m_{\mathcal{K}}(W^m_{\mathcal{K}'}(L))$ bounds a Seifert surface F_m in a regular neighborhood of $W_{\mathcal{K}'}(K_m)$, it follows that the components of $W^m_{\mathcal{K}}(W^m_{\mathcal{K}'}(L))$ bound m mutually disjoint Seifert surfaces $F_1, \ldots, F_m$.

(ii) We may suppose that $i = m-1$ and $j = m$ without loss of generality. $K_1 \cup \cdots \cup K_{m-2}$ being a boundary link, it bounds a disjoint union of surfaces $E = E_1 \cup \cdots \cup E_{m-2}$. Since $K_{m-1} \cup K_m$ is null-homotopic in $S^3 \setminus (K_1 \cup \cdots \cup K_{m-2})$, there is a band presentation of $K_{m-1} \cup K_m$ such that the base link is disjoint from E and such that the intersections of each band and E are ribbon singularities. By tubing the surfaces E_i suitably at the endpoints of theses singularities, we obtain a union of mutually disjoint surfaces $F_1, \ldots, F_{m-2}$ such that $F_i = \partial K_i$ and $F_i \cap (K_{m-1} \cup K_m) = \emptyset$ for all $1 \leq i \leq m-2$. Since the $(m-1)^{\text{th}}$ and m^{th} components of $W^{m-1}_{\mathcal{K}}(W^m_{\mathcal{K}'}(L))$ bound a disjoint union $F_{m-1} \cup F_m$ of Seifert surfaces in a regular neighborhood of $K_{m-1} \cup K_m$, it follows that the components of $W^{m-1}_{\mathcal{K}}(W^m_{\mathcal{K}'}(L))$ bound m mutually disjoint Seifert surfaces $F_1, \ldots, F_m$. $\square$

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